

THE BICROSSPRODUCT MONOIDAL HOM-HOPF ALGEBRA AND DRINFELD DOUBLE

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ABSTRACT. Let (B, β) and (H, α) be monoidal Hom-Hopf algebras, and (B, β) be a left (H, α) -module algebra, (H, α) be a right (B, β) -comodule coalgebra. In this paper we will construct the bicrossproduct $(B \# H, \beta \# \alpha)$, which generalize the bicrossproduct defined on usual Hopf algebras by S. Majid. Then a class of bicrossproduct monoidal Hom-Hopf algebras $(H * H^{op}, \alpha \otimes \alpha)$ can be constructed. Furthermore we will construct Drinfeld double $(H^{op} \bowtie H^*, \alpha \otimes (\alpha^{-1})^*)$ and prove that there exists a quasitriangular structure R on it satisfying the quantum Hom-Yang-Baxter equations.

0. INTRODUCTION

S. Majid in [1] firstly introduced the notion of bicrossproduct when searching for the self-dual algebraic structures and the non-commutative and non-cocommutative Hopf algebras. Since then many important works have been conducted on it. For example in [2], it was shown that the bicrossproduct structure is one-one corresponding to the double cross product structure, and in [7], the authors considered the quasitriangular structure on the bicrossproduct Hopf algebra. The quasitriangular structure R turned out to admit a very balanced and elegant decomposition.

Hom-algebras in [4] was introduced for the first time. Here the associativity in the usual algebras is replaced by the Hom-associativity and the unit no longer exists, replaced by a weak unit. Dually in [5] the notion of Hom-coalgebra was constructed. Then the concept of Hom-bialgebra and Hom-Hopf algebra were naturally developed as an generalization of the ordinary Hopf algebras. In [6], the authors illustrated the Hom-structure in the monoidal category approach, and introduced the monoidal Hom-Hopf algebras, which were slightly different from the previous Hom-algebras and Hom-coalgebras.

In [9], the authors defined the smash product on monoidal Hom-Hopf algebras, and then in [10], the smash product was used to construct Radford's biproduct on monoidal Hom-Hopf algebras. In [8], the author introduced the quasitriangular

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Hom-bialgebras(not monoidal Hom-bialgebras), as a generalization of the ordinary quasitriangular bialgebras and the quantum Hom-Yang-Baxter equation(QHYBE) of the form,

$$R^{12}(R^{13}R^{23}) = (R^{13}R^{23})R^{12}$$

and

$$(R^{12}R^{13})R^{23} = R^{23}(R^{13}R^{12}).$$

Motivated by these ideas, in this paper, we will construct the bicrossproduct on monoidal Hom-Hopf algebras and then in the framework of monoidal Hom-Hopf algebras, we will consider the Drinfeld double which could be deduced from the bicrossproduct Hopf algebra and the quasitriangular structure.

This paper is organized as follows. In section 1, we will recall the definitions and results of monoidal Hom-Hopf algebras, such as a monoidal Hom-algebra, a monoidal Hom-coalgebra, a monoidal Hom-module, a monoidal Hom-comodule and the Hom-smash products. In section 2, we will introduce the notion of bicrossproduct $(B\#H, \beta\#\alpha)$, and we will give the necessary and sufficient conditions for $(B\#H, \beta\#\alpha)$ to form a monoidal Hom-Hopf algebra, generalizing the bicrossproduct defined in [1]. An example will be given at the end of this section. In section 3, we will construct a class of bicrossproduct monoidal Hom-Hopf algebras. In Section 4, we will construct the Drinfeld double from the bicrossproduct monoidal Hopf algebra and obtain its quasitriangular structure. We also prove that it is a solution of quantum Hom-Yang-Baxter equation.

Throughout this article, all the vector spaces, tensor product and homomorphisms are over a fixed field k . We use the Sweedler's notation for the terminologies on coalgebras. For a coalgebra C , we write comultiplication $\Delta(c) = \sum c_1 \otimes c_2$ for any $c \in C$.

1. PRELIMINARIES

Let $\mathcal{M}_k = (\mathcal{M}_k, \otimes, k, a, l, r)$ be the category of k -modules. Now from this category, we could construct a new monoidal category $\mathcal{H}(\mathcal{M}_k)$. The objects of $\mathcal{H}(\mathcal{M}_k)$ are pairs (M, μ) , where $M \in \mathcal{M}_k$ and $\mu \in \text{Aut}_k(M)$. Any morphism $f : (M, \mu) \rightarrow (N, \nu)$ in $\mathcal{H}(\mathcal{M}_k)$ is a k -linear map from M to N such that $\nu \circ f = f \circ \mu$. For any objects (M, μ) and (N, ν) in $\mathcal{H}(\mathcal{M}_k)$, the monoidal structure is given by

$$(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu),$$

and the unit is (k, id_k) .

Generally speaking, all Hom-structure are objects in the monoidal category $\tilde{\mathcal{H}}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (k, id_k), \tilde{a}, \tilde{l}, \tilde{r})$ as introduced in [6], where the associativity

constraint \tilde{a} is given by the formula

$$\tilde{a}_{M,N,L} = a_{M,N,L} \circ ((\mu \otimes id) \otimes \lambda^{-1}) = (\mu \otimes (id \otimes \lambda^{-1})) \circ a_{M,N,L}$$

for any objects (M, μ) , (N, ν) , (L, λ) in $\mathcal{H}(\mathcal{M}_k)$. And the unit constraints \tilde{l} and \tilde{r} are defined by

$$\tilde{l}_M = \mu \circ l_M = l_M \circ (id \otimes \mu), \quad \tilde{r}_M = \mu \circ r_M = r_M \circ (\mu \otimes id).$$

The category $\tilde{\mathcal{H}}(\mathcal{M}_k)$ is called the Hom-category associated to the monoidal category \mathcal{M}_k . In what follows, we will recall the definitions in [6] on the monoidal Hom-associative algebras, monoidal Hom-coassociative coalgebras, monoidal Hom-modules and monoidal Hom-comodules.

Recall from [6], a unital monoidal Hom-associative algebra is an object (A, α) in the category $\tilde{\mathcal{H}}(\mathcal{M}_k)$ together with an element $1_A \in A$ and a linear map $m : A \otimes A \rightarrow A$, $a \otimes b \mapsto ab$ such that

$$\begin{aligned} \alpha(a)(bc) &= (ab)\alpha(c), \quad a1_A = \alpha(a) = 1_A a, \\ \alpha(ab) &= \alpha(a)\alpha(b), \quad \alpha(1_A) = 1_A, \end{aligned}$$

for all $a, b, c \in A$.

In the setting of Hopf algebras, m is called the Hom-multiplication, α is the twisting automorphism, and 1_A is the unit. Let (A, α) and (A', α') be two monoidal Hom-algebras. A Hom-algebra map $f : (A, \alpha) \rightarrow (A', \alpha')$ is a linear map such that $f \circ \alpha = \alpha' \circ f$, $f(ab) = f(a)f(b)$ and $f(1_A) = 1_{A'}$.

Obviously (A^{op}, α) is also a Hom-algebra.

Recall from [6], a counital monoidal Hom-coassociative coalgebra is an object (C, γ) in the category $\tilde{\mathcal{H}}(\mathcal{M}_k)$ together with linear maps $\Delta : C \rightarrow C \otimes C$, $c \mapsto c_1 \otimes c_2$ and $\varepsilon : C \rightarrow k$ such that

$$\begin{aligned} \sum \gamma^{-1}(c_1) \otimes \Delta(c_2) &= \sum \Delta(c_1) \otimes \gamma^{-1}(c_2), \quad \sum c_1 \varepsilon(c_2) = \sum \varepsilon(c_1) c_2 = \lambda^{-1}(c), \\ \Delta(\gamma(c)) &= \sum \gamma(c_1) \otimes \gamma(c_2), \quad \varepsilon \gamma(c) = \varepsilon(c), \end{aligned}$$

for all $c \in C$.

Let (C, γ) and (C', γ') be two monoidal Hom-coalgebras. A Hom-coalgebra map $f : (C, \gamma) \rightarrow (C', \gamma')$ is a linear map such that $f \circ \gamma = \gamma' \circ f$, $\Delta \circ f = (f \otimes f) \circ \Delta$ and $\varepsilon \circ f = \varepsilon$.

Recall from [6], a monoidal Hom-bialgebra $H = (H, \alpha, m, 1_H, \Delta, \varepsilon)$ is a bialgebra in the category $\tilde{\mathcal{H}}(\mathcal{M}_k)$ if $(H, \alpha, m, 1_H)$ is a monoidal Hom-algebra and $(H, \alpha, \Delta, \varepsilon)$ is a monoidal Hom-coalgebra such that Δ and ε are Hom-algebra maps, that is, for any $g, h \in H$,

$$\Delta(gh) = \Delta(g)\Delta(h), \quad \Delta(1_H) = 1_H \otimes 1_H,$$

$$\varepsilon(gh) = \varepsilon(g)\varepsilon(h), \quad \varepsilon(1_H) = 1.$$

A monoidal Hom-bialgebra (H, α) is called a monoidal Hom-Hopf algebra if there exists a linear map $S : H \rightarrow H$ (the antipode) such that

$$S \circ \alpha = \alpha \circ S, \quad \sum S(h_1)h_2 = \varepsilon(h)1_H = \sum h_1S(h_2).$$

Just as in the case of Hopf algebras, the antipode of monoidal Hom-Hopf algebras is a morphism of Hom-anti-algebras and Hom-anti-coalgebras.

When (H, α) is a finite-dimensional monoidal Hom-Hopf algebra, then $(H^*, (\alpha^{-1})^*)$ is also a monoidal Hom-Hopf algebra, where $(\alpha^{-1})^* : H^* \rightarrow H^*$, $f \mapsto f \circ \alpha^{-1}$, and the antipode is S_H^* defined similarly as $(\alpha^{-1})^*$.

Recall from [6], let (A, α) be a monoidal Hom-algebra. A left (A, α) -Hom-module is an object (M, μ) in $\tilde{\mathcal{H}}(\mathcal{M}_k)$ together with a linear map $\varphi : A \otimes M \rightarrow M$, $a \otimes m \mapsto am$ such that

$$\alpha(a)(bm) = (ab)\mu(m), \quad 1_A m = \mu(m),$$

$$\mu(am) = \alpha(a)\mu(m),$$

for all $a, b \in A$ and $m \in M$.

Similarly we can define the right (A, α) -Hom-modules. Let (M, μ) and (N, ν) be two left (A, α) -Hom-modules, then a linear map $f : M \rightarrow N$ is called left A -module map if $f(am) = af(m)$ for any $a \in A$, $m \in M$ and $f \circ \mu = \nu \circ f$.

Recall from [6], let (C, γ) be a monoidal Hom-coalgebra. A right (C, γ) -Hom-comodule is an object (M, μ) in $\tilde{\mathcal{H}}(\mathcal{M}_k)$ together with a linear map $\rho_M : M \rightarrow M \otimes C$, $m \mapsto m_{(0)} \otimes m_{(1)}$ such that

$$\sum \mu^{-1}(m_{(0)}) \otimes \Delta(m_{(1)}) = \sum \rho_M(m_{(0)}) \otimes \gamma^{-1}(m_{(1)}), \quad \sum \varepsilon(m_{(1)})m_{(0)} = \mu^{-1}(m),$$

$$\rho_M(\mu(m)) = \sum \mu(m_{(0)}) \otimes \gamma(m_{(1)}),$$

for all $m \in M$.

Let (M, μ) and (N, ν) be two right (C, γ) -Hom-comodules, then a linear map $g : M \rightarrow N$ is called right C -comodule map if $g \circ \mu = \nu \circ g$ and $\rho_N(g(m)) = (g \otimes id)\rho_M(m)$ for any $m \in M$.

Recall from [9]: let (H, α) be a monoidal Hom-bialgebra, and (B, β) be a monoidal Hom-algebra. We say (B, β) is a left (H, α) -Hom-module algebra if (B, β) is a left (H, α) -Hom-module with the action \cdot and satisfies

$$h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b), \quad h \cdot 1_B = \varepsilon(h)1_B,$$

for any $a, b \in B$ and $h \in H$.

Recall from [9]: let (H, α) be a monoidal Hom-bialgebra, and (B, β) be a monoidal Hom-coalgebra. We say (B, β) is a left (H, α) -Hom-module coalgebra if (B, β) is a left (H, α) -Hom-module with the action \cdot and satisfies

$$\Delta(h \cdot a) = \sum (h_1 \cdot a_1)(h_2 \cdot a_2), \quad \varepsilon_B(h \cdot a) = \varepsilon_H(h)\varepsilon_B(a),$$

for any $a \in B$ and $h \in H$.

Recall from [9], let (B, β) be a left (H, α) -Hom-module algebra. The Hom-smash product $(B \# H, \beta \# \alpha)$ of (B, β) and (H, α) is defined as follows:

- (a) $B \# H = B \otimes H$ as a vector space,
- (b) For any $a, b \in B$ and $h, k \in H$

$$(a \# h)(b \# k) = \sum a(h_1 \cdot \beta^{-1}(b)) \# \alpha(h_2)k.$$

Then $(B \# H, \beta \# \alpha)$ is a monoidal Hom-associative algebra with the unit $1_B \# 1_H$.

2. THE BICROSSPRODUCT CONSTRUCTION

In this section, we will construct the bicrossproduct of two monoidal Hom-Hopf algebras. Firstly dual to the concept of Hom-module algebra, we have the definition of Hom-comodule coalgebra as follows:

Recall from [10]: let (H, α) be a monoidal Hom-bialgebra, and (C, γ) be a monoidal Hom-coalgebra. We say (C, γ) is a right (H, α) -Hom-comodule coalgebra, if (C, γ) is a right (H, α) -Hom-comodule and with the coaction $\rho(c) = \sum c_{(0)} \otimes c_{(1)}$ and

- (a) $\sum c_{(0)1} \otimes c_{(0)2} \otimes c_{(1)} = \sum c_{1(0)} \otimes c_{2(0)} \otimes c_{1(1)}c_{2(1)},$
- (b) $\sum \varepsilon(c_{(0)})c_1 = \varepsilon(c)1_H.$

Now we have a dual version of Proposition 3.2 in [10].

Proposition 2.1. *Let (H, α) be a monoidal Hom-coalgebra and (B, β) be a monoidal Hom-bialgebra. Assume that (H, α) is a right monoidal (B, β) -Hom comodule coalgebra, then $(B \# H, \beta \# \alpha)$ is a monoidal Hom-coalgebra with the comultiplication and counit defined as follows:*

$$\Delta(a \# h) = \sum a_1 \# \alpha(h_{1(0)}) \otimes \beta^{-1}(a_2)h_{1(1)} \# h_2, \quad (2.1)$$

$$\varepsilon(a \# h) = \varepsilon_B(a)\varepsilon_H(h), \quad (2.2)$$

where $B \# H = B \otimes H$ as a vector space, and we use the same notion as in the Hom-smash product.

Proof. Firstly, for any $a \# h \in B \# H$,

$$\begin{aligned}
& \sum \Delta((a \# h)_1) \otimes (\beta^{-1} \# \alpha^{-1})((a \# h)_2) \\
&= \sum \Delta(a_1 \# \alpha(h_{1(0)})) \# (\beta^{-1} \otimes \alpha^{-1})(\beta^{-1}(a_2)h_{1(1)} \# h_2) \\
&= \sum a_{11} \# \alpha(\alpha(h_{1(0)})_{1(0)}) \otimes \beta^{-1}(a_{12})\alpha(h_{1(0)})_{1(1)} \# \alpha(h_{1(0)})_2 \\
&\quad \otimes \beta^{-2}(a_2)\beta^{-1}(h_{1(1)}) \# \alpha^{-1}(h_2) \\
&= \sum \beta^{-1}(a_1) \# \alpha(\alpha(h_{11(0)}))_{(0)} \otimes \beta^{-1}(a_{21})\alpha(h_{11(0)})_{(1)} \# \alpha(h_{12(0)}) \\
&\quad \otimes \beta^{-1}(a_{22})\beta^{-1}(h_{11(1)}h_{12(1)}) \# \alpha^{-1}(h_2) \\
&= \sum \beta^{-1}(a_1) \# \alpha(\alpha(\alpha^{-1}(h_1)_{(0)}))_{(0)} \otimes \beta^{-1}(a_{21})\alpha(\alpha^{-1}(h_1)_{(0)})_{(1)} \\
&\quad \# \alpha(h_{21(0)}) \otimes \beta^{-1}(a_{22})\beta^{-1}(\alpha^{-1}(h_1)_{(1)}h_{21(1)}) \# h_{22} \\
&\quad \text{(by coassociativity)} \\
&= \sum \beta^{-1}(a_1) \# \alpha(\alpha(\alpha^{-1}(h_1)_{(0)}))_{(0)} \otimes \beta^{-1}(a_{21})\alpha(\alpha^{-1}(h_1)_{(0)})_{(1)} \\
&\quad \# \alpha(h_{21(0)}) \otimes \beta^{-1}(a_{22})\beta^{-1}(\beta^{-1}(h_{1(1)})h_{21(1)}) \# h_{22} \\
&= \sum \beta^{-1}(a_1) \# \alpha(h_{1(0)(0)}) \otimes \beta^{-1}(a_{21})h_{1(0)(1)} \\
&\quad \# \alpha(h_{21(0)}) \otimes \beta^{-1}(a_{22})\beta^{-1}(\beta^{-1}(h_{1(1)})h_{21(1)}) \# h_{22} \\
&= \sum \beta^{-1}(a_1) \# h_{1(0)} \otimes \beta^{-1}(a_{21})h_{1(1)1} \\
&\quad \# \alpha(h_{21(0)}) \otimes \beta^{-1}(a_{22})\beta^{-1}(h_{1(1)2}h_{21(1)}) \# h_{22}, \\
&\quad \text{(by comodule condition)}
\end{aligned}$$

On the other hand

$$\begin{aligned}
& \sum (\beta^{-1} \# \alpha^{-1})((a \# h)_1) \otimes \Delta((a \# h)_2) \\
&= \sum (\beta^{-1} \# \alpha^{-1})(a_1 \# \alpha(h_{1(0)})) \otimes \Delta(\beta^{-1}(a_2)h_{1(1)} \# h_2) \\
&= \sum \beta^{-1}(a_1) \# h_{1(0)} \otimes \beta^{-1}(a_{21})h_{1(1)1} \# \alpha(h_{21(0)}) \\
&\quad \otimes \beta^{-1}(\beta^{-1}(a_{22})h_{1(1)2})h_{21(1)} \# h_{22} \\
&= \sum \beta^{-1}(a_1) \# h_{1(0)} \otimes \beta^{-1}(a_{21})h_{1(1)1} \# \alpha(h_{21(0)}) \\
&\quad \otimes (\beta^{-2}(a_{22})\beta^{-1}(h_{1(1)2}))h_{21(1)} \# h_{22} \\
&= \sum \beta^{-1}(a_1) \# h_{1(0)} \otimes \beta^{-1}(a_{21})h_{1(1)1} \# \alpha(h_{21(0)}) \otimes \\
&\quad \beta^{-1}(a_{22})(\beta^{-1}(h_{1(1)2})\beta^{-1}(h_{21(1)})) \# h_{22}.
\end{aligned}$$

Hence we have the Hom-coassociativity.

Secondly

$$\begin{aligned}
& \sum (\varepsilon_B \otimes \varepsilon_H \otimes id \otimes id)(a_1 \# \alpha(h_{1(0)}) \otimes \beta^{-1}(a_2)h_{1(1)} \otimes h_2) \\
&= \sum \varepsilon_B(a_1)\varepsilon_H(\alpha(h_{1(0)}))\beta^{-1}(a_2)h_{1(1)} \otimes h_2 \\
&= \sum \beta^{-2}(a)1_B \# \varepsilon_H(h_1)h_2 \\
&= \sum \beta^{-1}(a) \# \alpha^{-1}(h).
\end{aligned}$$

and

$$\begin{aligned}
& \sum (id \otimes id \otimes \varepsilon_B \otimes \varepsilon_H)(a_1 \# \alpha(h_{1(0)}) \otimes \beta^{-1}(a_2)h_{1(1)} \otimes h_2) \\
&= \sum a_1 \# \alpha(h_{1(0)})\varepsilon_B(\beta^{-1}(a_2))\varepsilon_B(h_{1(1)})\varepsilon_H(h_2) \\
&= \sum \beta^{-1}(a) \# h_1 \varepsilon_H(h_2) \\
&= \sum \beta^{-1}(a) \# \alpha^{-1}(h)
\end{aligned}$$

We have that $(B \# H, \beta \# \alpha)$ is a monoidal Hom-coalgebra. The proof is completed. \square

Here we will call $(B \# H, \beta \# \alpha)$ Hom-smash coproduct as in [10].

Before the next proposition, we have the following identities by the coassociativity:

$$\sum h_{11} \otimes h_{12} \otimes h_{211} \otimes h_{212} \otimes h_{22} = \sum \alpha^{-1}(h_1) \otimes \alpha^2(h_{2111}) \otimes \alpha(h_{2112}) \otimes h_{212} \otimes h_{22} \quad (2.3)$$

$$\sum h_1 \otimes h_{211} \otimes h_{212} \otimes h_{22} = \sum \alpha(h_{11}) \otimes \alpha^{-1}(h_{12}) \otimes \alpha^{-1}(h_{21}) \otimes h_{22} \quad (2.4)$$

Proposition 2.2. *Let (B, β) and (H, α) be two monoidal Hom-Hopf algebras. $(B \# H, \beta \# \alpha)$ is a Hom-smash product and also a Hom-smash coproduct defined as above, then $(B \# H, \beta \# \alpha)$ is a Hom-bialgebra if and only if*

- (a) $\Delta(h \cdot b) = \sum \alpha(h_{1(0)}) \cdot b_1 \otimes \beta(h_{1(1)})(\alpha^{-1}(h_2) \cdot \beta^{-1}(b_2)),$
- (b) $\varepsilon_B(h \cdot b) = \varepsilon_H(h)\varepsilon_B(b),$
- (c) $\rho(1_H) = 1_H \otimes 1_B,$
- (d) $\sum h_{2(0)} \otimes (h_1 \cdot b)\beta^2(h_{2(1)}) = \sum h_{1(0)} \otimes \beta^2(h_{1(1)})(h_2 \cdot b),$
- (e) $\sum (hk)_{(0)} \otimes (hk)_{(1)} = \sum \alpha(h_{1(0)})k_{(0)} \otimes \beta(h_{1(1)})(\alpha^{-1}(h_2) \cdot \beta^{-1}(k_{(1)})).$

Proof. If $(B \# H, \beta \# \alpha)$ is a monoidal Hom-bialgebra, by $\Delta(1 \# 1) = 1 \# 1 \otimes 1 \# 1$, we have

$$\sum 1 \# \alpha(1_0) \otimes \beta(1_{(1)}) \# 1 = 1 \# 1 \otimes 1 \# 1.$$

Then

$$\sum 1 \# \alpha(1)_0 \otimes \alpha(1)_{(1)} \# 1 = 1 \# 1 \otimes 1 \# 1,$$

hence

$$\sum 1_{(0)} \otimes 1_{(1)} = 1_H \otimes 1_B.$$

And by $\varepsilon((a\#h)(b\#k)) = \varepsilon(a\#h)\varepsilon(b\#k)$ we have

$$\sum \varepsilon(a(h_1 \cdot \beta^{-1}(b))\#\alpha(h_2)k) = \varepsilon_B(a)\varepsilon_B(b)\varepsilon_H(h)\varepsilon_H(k).$$

Then

$$\sum \varepsilon_B(a)\varepsilon_B(h_1 \cdot \beta^{-1}(b))\varepsilon_H(h_2k) = \varepsilon_B(a)\varepsilon_B(b)\varepsilon_H(h)\varepsilon_H(k).$$

Hence

$$\varepsilon_B(h \cdot b) = \varepsilon_H(h)\varepsilon_B(b).$$

By

$$\begin{aligned} & \Delta((a\#h)(b\#k)) \\ &= \sum \Delta(a(h_1 \cdot \beta^{-1}(b))\#\alpha(h_2)k) \\ &= \sum a_1(h_1 \cdot \beta^{-1}(b))_1\#\alpha((\alpha(h_{21})k_1)_{(0)}) \\ & \quad \otimes \beta^{-1}(a_2(h_1 \cdot \beta^{-1}(b))_2)(\alpha(h_{21})k_1)_{(1)}\#\alpha(h_{22})k_2, \end{aligned}$$

and

$$\begin{aligned} & \Delta(a\#h)\Delta(b\#k) \\ &= \sum [a_1\#\alpha(h_{1(0)}) \otimes \beta^{-1}(a_2)h_{1(1)}\#h_2][b_1\#\alpha(k_{1(0)}) \otimes \beta^{-1}(b_2)k_{1(1)}\#k_2] \\ &= \sum a_1(\alpha(h_{1(0)})_1 \cdot \beta^{-1}(b_1))\#\alpha(\alpha(h_{1(0)})_2)\alpha(k_{1(0)}) \\ & \quad \otimes (\beta^{-1}(a_2)h_{1(1)})(h_{21} \cdot \beta^{-1}(\beta^{-1}(b_2)k_{1(1)}))\#\alpha(h_{22})k_2 \\ &= \sum a_1(\alpha(h_{1(0)1}) \cdot \beta^{-1}(b_1))\#\alpha^2(h_{1(0)2})\alpha(k_{1(0)}) \\ & \quad \otimes (\beta^{-1}(a_2)h_{1(1)})(h_{21} \cdot \beta^{-2}(b_2)\beta^{-1}(k_{1(1)}))\#\alpha(h_{22})k_2, \end{aligned}$$

we have

$$\begin{aligned} & \sum a_1(h_1 \cdot \beta^{-1}(b))_1\#\alpha((\alpha(h_{21})k_1)_{(0)}) \\ & \quad \otimes \beta^{-1}(a_2(h_1 \cdot \beta^{-1}(b))_2)(\alpha(h_{21})k_1)_{(1)}\#\alpha(h_{22})k_2 \\ &= \sum a_1(\alpha(h_{1(0)1}) \cdot \beta^{-1}(b_1))\#\alpha^2(h_{1(0)2})\alpha(k_{1(0)}) \\ & \quad \otimes (\beta^{-1}(a_2)h_{1(1)})(h_{21} \cdot \beta^{-2}(b_2)\beta^{-1}(k_{1(1)}))\#\alpha(h_{22})k_2. \end{aligned} \tag{2.5}$$

Let $a = 1$ and $k = 1$

$$\begin{aligned} & \sum \beta((h_1 \cdot \beta^{-1}(b))_1)\#\alpha((\alpha^2(h_{21}))_{(0)}) \otimes (h_1 \cdot \beta^{-1}(b))_2\alpha^2(h_{21})_{(1)}\#\alpha^2(h_{22}) \\ &= \sum \alpha^2(h_{1(0)1}) \cdot b_1\#\alpha^3(h_{1(0)2}) \otimes \beta(h_{1(1)})(h_{21} \cdot \beta^{-1}(b_2))\#\alpha^2(h_{22}). \end{aligned} \tag{2.6}$$

Applying $id \otimes \varepsilon_H \otimes id \otimes \varepsilon_H$ to both sides of (2.6), we have

$$\begin{aligned} & \sum \beta((h_1 \cdot \beta^{-1}(b))_1) \varepsilon_H(h_{21}) \otimes (h_1 \cdot \beta^{-1}(b))_2 1_H \varepsilon_H(h_{22}) \\ &= \sum \alpha^2(h_{1(0)1}) \cdot b_1 \varepsilon_H(h_{1(0)2}) \otimes \beta(h_{1(1)})(h_{21} \cdot \beta^{-1}(b_2)) \varepsilon_H(h_{22}), \end{aligned}$$

so

$$\sum (h \cdot b)_1 \otimes (h \cdot b)_2 = \sum \alpha(h_{1(0)} \cdot b_1) \otimes \beta(h_{1(1)})(\alpha^{-1}(h_2) \cdot \beta^{-1}(b_2)).$$

Applying $\varepsilon_B \otimes id \otimes id \otimes \varepsilon_H$ to both sides of (2.6), we have

$$\begin{aligned} & \sum \alpha^2(h_{2(0)}) \otimes \beta^{-1}(h_1 \cdot \beta^{-1}(b)) \beta(h_{2(1)}) \\ &= \sum \alpha^2(h_{1(0)}) \otimes \beta(h_{1(1)}) \beta^{-1}(h_2 \cdot \beta^{-1}(b)), \end{aligned}$$

so

$$\sum h_{2(0)} \otimes (h_1 \cdot b) \beta^2(h_{2(1)}) = \sum h_{1(0)} \otimes \beta^2(h_{1(1)})(h_2 \cdot b).$$

Let $a = b = 1$, then

$$\begin{aligned} & \sum 1 \# \varepsilon_H(h_1) \alpha((\alpha(h_{21})k_1)_{(0)}) \otimes \beta(\alpha(h_{21})k_1)_{(1)} \# \alpha(h_{22})k_2 \\ &= \sum 1 \# \varepsilon_H(h_{1(0)1}) \alpha^2(h_{1(0)2}) \alpha(k_{1(0)}) \otimes \beta(h_{1(1)})(h_{21} \cdot k_{1(1)}) \# \alpha(h_{22})k_2, \end{aligned}$$

then

$$\begin{aligned} & \sum 1 \# \alpha((h_1 k_1)_{(0)}) \otimes \beta((h_1 k_1)_{(1)}) \# \alpha(h_2)k_2 \\ &= \sum 1 \# \alpha(h_{1(0)}) \alpha(k_{1(0)}) \otimes \beta(h_{1(1)})(h_{21} \cdot k_{1(1)}) \# \alpha(h_{22})k_2. \end{aligned} \tag{2.7}$$

Applying $\varepsilon_B \otimes id \otimes id \otimes \varepsilon_H$ to both sides of the above equation, we have

$$\sum (hk)_{(0)} \otimes (hk)_{(1)} = \sum \alpha(h_{1(0)})k_{(0)} \otimes \beta(h_{1(1)})(\alpha^{-1}(h_2) \cdot \beta^{-1}(k_{(1)})).$$

Conversely if the condition (a), (b), (c) (d) and (e) are satisfied,

$$\begin{aligned} & \Delta((a \# h)(b \# k)) \\ &= \sum \Delta(a(h_1 \cdot \beta^{-1}(b)) \# \alpha(h_2)k) \\ &= \sum a_1(h_1 \cdot \beta^{-1}(b))_1 \# \alpha((\alpha(h_{21})k_1)_{(0)}) \\ & \quad \otimes \beta^{-1}(a_2(h_1 \cdot \beta^{-1}(b))_2)(\alpha(h_{21})k_1)_{(1)} \# \alpha(h_{22})k_2 \\ &= \sum a_1(\alpha(h_{11(0)}) \cdot \beta^{-1}(b_1)) \# \alpha((\alpha(h_{21})k_1)_{(0)}) \\ & \quad \otimes \beta^{-1}(a_2(\beta(h_{1(1)})(\alpha^{-1}(h_{12}) \cdot \beta^{-2}(b_2))))(\alpha(h_{21})k_1)_{(1)} \# \alpha(h_{22})k_2 \text{ by (a)} \\ &= \sum a_1(\alpha(h_{11(0)}) \cdot \beta^{-1}(b_1)) \# \alpha(\alpha(h_{211(0)}))k_{1(0)} \\ & \quad \otimes [\beta^{-1}(a_2)(h_{11(1)})(\alpha^{-2}(h_{12}) \cdot \beta^{-3}(b_2)))] \\ & \quad [\beta(\beta(h_{211(1)}))(\alpha^{-1}(\alpha(h_{212})) \cdot \beta^{-1}(k_{1(1)}))] \# \alpha(h_{22})k_2 \end{aligned}$$

$$\begin{aligned}
&= \sum a_1(\alpha(h_{11(0)}) \cdot \beta^{-1}(b_1)) \# \alpha^3(h_{211(0)}) \alpha(k_{1(0)}) \\
&\quad \otimes [\beta^{-1}(a_2)(h_{11(1)})(\alpha^{-2}(h_{12}) \cdot \beta^{-3}(b_2))] \\
&\quad [\beta^2(h_{211(1)})(h_{212} \cdot \beta^{-1}(k_{1(1)}))] \# \alpha(h_{22}) k_2 \text{ by (e)} \\
&= \sum a_1(\alpha(h_{11(0)}) \cdot \beta^{-1}(b_1)) \# \alpha^3(h_{212(0)}) \alpha(k_{1(0)}) \\
&\quad \otimes [\beta^{-1}(a_2)(h_{11(1)})(\alpha^{-2}(h_{12}) \cdot \beta^{-3}(b_2))] \\
&\quad [(h_{211} \cdot \beta^{-1}(k_{1(1)})) \beta^2(h_{212(1)})] \# \alpha(h_{22}) k_2 \text{ by (d)} \\
&= \sum a_1(h_{1(0)} \cdot \beta^{-1}(b_1)) \# \alpha^3(h_{212(0)}) \alpha(k_{1(0)}) \\
&\quad \otimes [\beta^{-1}(a_2)(\beta^{-1}(h_{1(1)})(h_{2111} \cdot \beta^{-3}(b_2)))] \\
&\quad [(\alpha(h_{2112}) \cdot \beta^{-1}(k_{1(1)})) \beta^2(h_{212(1)})] \# \alpha(h_{22}) k_2 \text{ by (2.3)} \\
&= \sum a_1(h_{1(0)} \cdot \beta^{-1}(b_1)) \# \alpha^3(h_{212(0)}) \alpha(k_{1(0)}) \\
&\quad \otimes [(\beta^{-2}(a_2) \beta^{-1}(h_{1(1)}))(\alpha(h_{2111}) \cdot \beta^{-2}(b_2))] \\
&\quad [(\alpha(h_{2112}) \cdot \beta^{-1}(k_{1(1)})) \beta^2(h_{212(1)})] \# \alpha(h_{22}) k_2 \\
&= \sum a_1(h_{1(0)} \cdot \beta^{-1}(b_1)) \# \alpha^3(h_{212(0)}) \alpha(k_{1(0)}) \\
&\quad \otimes [(\beta^{-2}(a_2) \beta^{-1}(h_{1(1)}))(h_{211} \cdot \beta^{-3}(b_2) \beta^{-2}(k_{1(1)}))] \beta^3(h_{212(1)}) \# \alpha(h_{22}) k_2 \\
&= \sum a_1(h_{1(0)} \cdot \beta^{-1}(b_1)) \# \alpha^3(h_{212(0)}) \alpha(k_{1(0)}) \\
&\quad \otimes (\beta^{-1}(a_2) h_{1(1)}) [(h_{211} \cdot \beta^{-3}(b_2) \beta^{-2}(k_{1(1)})) \beta^2(h_{212(1)})] \# \alpha(h_{22}) k_2 \\
&= \sum a_1(h_{1(0)} \cdot \beta^{-1}(b_1)) \# \alpha^3(h_{211(0)}) \alpha(k_{1(0)}) \\
&\quad \otimes (\beta^{-1}(a_2) h_{1(1)}) [\beta^2(h_{211(1)})(h_{212} \cdot \beta^{-3}(b_2) \beta^{-2}(k_{1(1)}))] \# \alpha(h_{22}) k_2 \text{ by (d)} \\
&= \sum a_1(\alpha(h_{11(0)}) \cdot \beta^{-1}(b_1)) \# \alpha^3(\alpha^{-1}(h_{12(0)})) \alpha(k_{1(0)}) \\
&\quad \otimes (\beta^{-1}(a_2) \beta(h_{11(1)})) [\beta(h_{12(1)}) \\
&\quad (\alpha^{-1}(h_{21}) \cdot \beta^{-3}(b_2) \beta^{-2}(k_{1(1)}))] \# \alpha(h_{22}) k_2 \text{ by (2.4)} \\
&= \sum a_1(\alpha(h_{11(0)}) \cdot \beta^{-1}(b_1)) \# \alpha^3(\alpha^{-1}(h_{12(0)})) \alpha(k_{1(0)}) \\
&\quad \otimes [\beta^{-1}(a_2)(h_{11(1)} h_{12(1)})] (h_{21} \cdot \beta^{-2}(b_2) \beta^{-1}(k_{1(1)})) \# \alpha(h_{22}) k_2 \\
&= \sum a_1(\alpha(h_{1(0)1}) \cdot \beta^{-1}(b_1)) \# \alpha^2(h_{1(0)2}) \alpha(k_{1(0)}) \\
&\quad \otimes (\beta^{-1}(a_2) h_{1(1)}) (h_{21} \cdot \beta^{-2}(b_2) \beta^{-1}(k_{1(1)})) \# \alpha(h_{22}) k_2 \text{ by comodule coalgebra} \\
&= \Delta(a \# h) \Delta(b \# k)
\end{aligned}$$

And it is easy to check that $\varepsilon((a \# h)(b \# k)) = \varepsilon(a \# h) \varepsilon(b \# k)$. Therefore $(B \# H, \beta \# \alpha)$ is a monoidal Hom-bialgebra. The proof is completed. \square

Proposition 2.3. *Let (B, β) and (H, α) be monoidal Hom-Hopf algebras, and we have the bicrossproduct monoidal Hom-bialgebra $(B \# H, \beta \# \alpha)$ defined as above. Define the antipode by*

$$S(a \# h) = \sum (1_B \# S_H(h_{(0)}))(S_B(\beta^{-2}(a)\beta^{-1}(h_{(1)}))\# 1_H), \quad (2.8)$$

then $(B \# H, \beta \# \alpha)$ is a monoidal Hom-Hopf algebra.

Proof. For any $a \# h \in B \# H$,

$$\begin{aligned} & \sum S(a_1 \# \alpha(h_{1(0)}))(\beta^{-1}(a_2)h_{1(1)} \# h_2) \\ &= \sum [(1_B \# S_H \alpha(h_{1(0)(0)}))(S_B(\beta^{-2}(a_1)\beta^{-1}\beta(h_{1(0)(1)}))\# 1_H)](\beta^{-1}(a_2)h_{1(1)} \# h_2) \\ &= \sum [(1_B \# S_H \alpha(h_{1(0)(0)}))(S_B(\beta^{-2}(a_1)h_{1(0)(1)})\# 1_H)](\beta^{-1}(a_2)h_{1(1)} \otimes h_2) \\ &= \sum (1_B \# S_H \alpha^2(h_{1(0)(0)}))[(S_B(\beta^{-2}(a_1)h_{1(0)(1)})\# 1_H)(\beta^{-2}(a_2)\beta^{-1}(h_{1(1)})\# \alpha^{-1}(h_2))] \\ &= \sum (1_B \# S_H \alpha^2(h_{1(0)(0)}))[(S_B(\beta^{-2}(a_1)h_{1(0)(1)})(\beta^{-2}(a_2)\beta^{-1}(h_{1(1)}))\# h_2] \\ &= \sum (1_B \# S_H \alpha^2(h_{1(0)(0)}))[(S_B(h_{1(0)(1)})S_B\beta^{-2}(a_1))(\beta^{-2}(a_2)\beta^{-1}(h_{1(1)}))\# h_2] \\ &= \sum (1_B \# S_H \alpha^2(h_{1(0)(0)}))[(S_B(h_{1(0)(1)})(S_B\beta^{-3}(a_1)\beta^{-3}(a_2)))h_{1(1)}\# h_2] \\ &= \sum (1_B \# S_H \alpha^2(h_{1(0)(0)}))[(S_B\beta(h_{1(0)(1)})h_{1(1)}\# h_2]\varepsilon_B(a) \\ &= \sum (1_B \# S_H \alpha(h_{1(0)}))[(S_B\beta(h_{1(1)1})\beta(h_{1(1)2})\# h_2]\varepsilon_B(a) \\ &= \sum (1_B \# S_H \alpha(h_{1(0)}))(\varepsilon_B(h_{1(1)}))1\# h_2\varepsilon_B(a) \\ &= \sum (1_B \# S_H(h_1)h_2)\varepsilon_B(a) \\ &= \sum (1_B \# 1_H)\varepsilon_B(a)\varepsilon(h). \end{aligned}$$

That is $S * id = \varepsilon$. Similarly we have $id * S = \varepsilon$. This completes the proof. \square

Remark 2.4. In the Proposition 2.2, if the module action is trivial, that is $h \cdot a = \varepsilon_H(h)\alpha(a)$, we have a bicrossproduct with monoidal Hom-smash coproduct and tensor product. If the comodule is trivial, that is $\rho(h) = \alpha^{-1}(h) \otimes 1$, we have a Hom-smash with monoidal Hom-smash product and tensor coproduct.

Next we will give an example.

Example 2.5. Let $B = \text{span}\{1, x\}$ over a fixed field k with $\text{char } k \neq 2$. β is a k -linear automorphism of B by

$$\beta(1_B) = 1_B, \quad \beta(x) = -x.$$

Define the multiplication on B by

$$1_B 1_B = 1_B, \quad 1_B x = -x, \quad x^2 = 0,$$

then (B, β) is a Hom-associative algebra;

the comultiplication and the counit by

$$\Delta(1_B) = 1_B \otimes 1_B, \quad \Delta(x) = (-x) \otimes 1 + 1 \otimes (-x),$$

$$\varepsilon(1_B) = 1, \quad \varepsilon(x) = 0,$$

$$S_B(1_B) = 1_B, \quad S_B(x) = -x,$$

then (B, β) is a monoidal Hom-Hopf algebra.

Let $H = \text{span}\{1, g | g^2 = 1\}$ be the group algebra. Then obviously (H, id) is a monoidal Hom-Hopf algebra.

Define the left action of H on B by $\cdot : H \otimes B \rightarrow B$ by

$$1_H \cdot 1_B = 1_B, \quad 1_H \cdot x = -x, \quad g \cdot 1_B = 1_B, \quad g \cdot x = x.$$

Then it is easy to check that (B, β) is a left (H, id) -module algebra.

Define the right coaction of B on H $\rho : H \rightarrow H \otimes B$ by

$$\rho(1_H) = 1_H \otimes 1_B, \quad \rho(g) = g \otimes 1_B,$$

then we can verify that (H, id) is a right (B, β) -comodule coalgebra, and the conditions in the above proposition can be satisfied. Hence we have a monoidal Hom-Hopf algebra $(B \# H, \beta \# id)$, where

$$\Delta(1_B \# 1_H) = 1_B \# 1_H \otimes 1_B \# 1_H, \quad \varepsilon(1_B \# 1_H) = 1,$$

$$\Delta(1_B \# g) = 1_B \# g \otimes 1_B \# g, \quad \varepsilon(1_B \# g) = 1,$$

$$\Delta(x \# 1_H) = (-x) \# 1_H \otimes 1_B \# 1_H, \quad \varepsilon(x \# 1_H) = 0,$$

$$\Delta(x \# g) = (-x) \# g \otimes 1_B \# g + 1_B \# g \otimes (-x) \# g, \quad \varepsilon(x \# g) = 0.$$

$$S(1_B \# 1_H) = 1_B \# 1_H, \quad S(1_B \# g) = 1_B \# g, \quad S(x \# 1_H) = x \otimes 1_H, \quad S(x \# g) = (-x) \# g.$$

As to the multiplication, it is the same as in the Example 3.7 in [10].

3. A CLASS OF MONOIDAL HOM-HOPF ALGEBRAS

In this subsection, we will construct a class of monoidal Hom-Hopf algebras as an application.

Lemma 3.1. *Let (H, α) be a monoidal Hom-Hopf algebra. Then define the left action of (H^{op}, α) on (H, α) by*

$$\cdot : H^{op} \otimes H \rightarrow H, \quad h \cdot a = \sum (S(h_1) \alpha^{-1}(a)) \alpha(h_2),$$

for any $h, a \in H$.

Then (H, α) is a left (H^{op}, α) -module algebra.

Proof. For any $h, k, a, b \in H$, firstly, $1 \cdot a = \alpha(a)$, and

$$\alpha(h \cdot a) = \sum (S\alpha(h_1)a)\alpha^2(h_2) = \alpha(h) \cdot \alpha(a).$$

Secondly

$$\begin{aligned} (hk) \cdot a &= \sum (S(h_1k_1)\alpha^{-1}(a))\alpha(h_2k_2) \\ &= \sum [(S(k_1)S(h_1))\alpha^{-1}(a)]\alpha(h_2k_2) \\ &= \sum [\alpha S(k_1)(S(h_1)\alpha^{-2}(a))]\alpha(h_2k_2) \\ &= \sum [(S(k_1)(\alpha^{-1}S(h_1)\alpha^{-3}(a)))\alpha(h_2)]\alpha^2(k_2) \\ &= \sum [\alpha S(k_1)((\alpha^{-1}S(h_1)\alpha^{-3}(a))h_2)]\alpha^2(k_2) \\ &= \sum \alpha(k) \cdot [(S(h_1)\alpha^{-2}(a))\alpha(h_2)] \\ &= \alpha(k) \cdot [h \cdot \alpha^{-1}(a)], \end{aligned}$$

Now (H, α) is a left (H^{op}, α) -module. Finally

$$\begin{aligned} \sum (h_1 \cdot a)(h_2 \cdot b) &= \sum ((S(h_{11})\alpha^{-1}(a))\alpha(h_{12}))((S(h_{21})\alpha^{-1}(b))\alpha(h_{22})) \\ &= \sum ((S(h_{11})\alpha^{-1}(a))\alpha(h_{12}))(S\alpha(h_{21})(\alpha^{-1}(b)h_{22})) \\ &= \sum [(S(h_{11})\alpha^{-1}(a))(h_{12}S(h_{21}))](b\alpha(h_{22})) \\ &= \sum [(S\alpha^{-1}(h_1)\alpha^{-1}(a))(\alpha(h_{211})S\alpha(h_{212}))](b\alpha(h_{22})) \\ &= \sum [(S(h_1)a)](bh_2) = (S(h_1)\alpha^{-1}(ab))\alpha(h_2) \\ &= h \cdot (ab), \end{aligned}$$

and

$$h \cdot 1 = \sum \alpha(S(h_1))\alpha(h_2) = \varepsilon(h)1.$$

This completes the proof. \square

Lemma 3.2. *Let (H, α) be a monoidal Hom-Hopf algebra. Define the right coaction of (H, α) on (H^{op}, α) by*

$$\rho : H^{op} \rightarrow H^{op} \otimes H, \quad \rho(h) = \sum \alpha(h_{12}) \otimes S(h_{11})\alpha^{-1}(h_2),$$

for any $h \in H$.

Then (H^{op}, α) is a right (H, α) -comodule coalgebra.

Proof. For any $h \in H$, firstly

$$\sum \alpha(h_{12})\varepsilon(h_{11})\varepsilon(h_2) = \alpha^{-1}(h),$$

and

$$\rho(\alpha(h)) = \sum \alpha(\alpha(h)_{12}) \otimes S(\alpha(h)_{11})\alpha^{-1}(\alpha(h)_2) = (\alpha \otimes \alpha)\rho(h).$$

Secondly

$$\begin{aligned}
& \sum \alpha^{-1}(\alpha(h_{12})) \otimes \Delta(S(h_{11})\alpha^{-1}(h_2)) \\
&= \sum h_{12} \otimes S(h_{112})\alpha^{-1}(h_{21}) \otimes S(h_{111})\alpha^{-1}(h_{22}) \\
&= \sum \alpha^2(h_{1122}) \otimes S\alpha(h_{1121})\alpha^{-1}(h_{12}) \otimes S(h_{111})\alpha^{-2}(h_2) \\
&= \sum \alpha^2(h_{1212}) \otimes S\alpha(h_{1211})h_{122} \otimes S\alpha^{-1}(h_{11})\alpha^{-2}(h_2) \\
&= \sum h_{(0)(0)} \otimes h_{(0)(1)} \otimes \alpha^{-1}(h_{(1)}).
\end{aligned}$$

Then (H^{op}, α) is a right (H, α) -comodule. Finally

$$\begin{aligned}
& \sum h_{1(0)} \otimes h_{2(0)} \otimes h_{1(1)}h_{1(1)} \\
&= \sum \alpha(h_{112}) \otimes \alpha(h_{212}) \otimes [S(h_{111})\alpha^{-1}(h_{12})][S(h_{211})\alpha^{-1}(h_{22})] \\
&= \sum \alpha(h_{112}) \otimes \alpha(h_{212}) \otimes [S(h_{111})(\alpha^{-2}(h_{12})\alpha^{-1}S(h_{211}))]h_{22} \\
&= \sum h_{12} \otimes \alpha(h_{212}) \otimes [S\alpha^{-1}(h_{11})(h_{2111}S(h_{2112}))]h_{22} \\
&= \sum h_{12} \otimes h_{21} \otimes S(h_{11})h_{22} \\
&= \sum \alpha(h_{121}) \otimes \alpha(h_{122}) \otimes S(h_{11})\alpha^{-1}(h_2) \\
&= \sum \Delta(\alpha(h_{12})) \otimes S(h_{11})\alpha^{-1}(h_2),
\end{aligned}$$

and

$$\sum \varepsilon(\alpha(h_{12}))S(h_{11})\alpha^{-1}(h_2) = \varepsilon(h)1.$$

This completes the proof. \square

Now we have the following result.

Proposition 3.3. *Let (H, α) be a monoidal Hom-Hopf algebra. The action $\cdot : H^{op} \otimes H \rightarrow H$ and coaction $\rho : H^{op} \rightarrow H^{op} \otimes H$ are defined as above. Then we have a bicrossproduct monoidal Hom-Hopf algebra $(H * H^{op}, \alpha * \alpha)$.*

Proof. For any $h, k, a, b \in H$, firstly

$$\begin{aligned}
\Delta(h \cdot a) &= \sum \Delta((S(h_1)\alpha^{-1}(a))\alpha(h_2)) \\
&= \sum (S(h_{12})\alpha^{-1}(a_1))\alpha(h_{21}) \otimes (S(h_{11})\alpha^{-1}(a_2))\alpha(h_{22}) \\
&= \sum \alpha(h_{21}) \cdot a_1 \otimes (S\alpha^{-1}(h_1)\alpha^{-1}(a_2))\alpha(h_{22}) \\
&= \sum \alpha(h_{12}) \cdot a_1 \otimes (S(h_{11})\alpha^{-1}(a_2))h_2 \\
&= \sum \alpha(h_{12}) \cdot a_1 \otimes S\alpha(h_{11})(\alpha^{-1}(a_2)\alpha^{-1}(h_2)) \\
&= \sum \alpha(h_{12}) \cdot a_1 \otimes S\alpha(h_{11})(\varepsilon(h_{21})\alpha^{-1}(a_2)h_{22}) \\
&= \sum \alpha^2(h_{112}) \cdot a_1 \otimes \alpha^2 S(h_{111})(((\alpha^{-2}(h_{12})S\alpha^{-2}(h_{21}))\alpha^{-2}(a_2))h_{22}) \\
&= \sum \alpha^2(h_{112}) \cdot a_1 \otimes \alpha^2 S(h_{111})((\alpha^{-1}(h_{12})(S\alpha^{-2}(h_{21})\alpha^{-3}(a_2)))h_{22}) \\
&= \sum \alpha^2(h_{112}) \cdot a_1 \otimes \alpha^2 S(h_{111})(h_{12}((S\alpha^{-2}(h_{21})\alpha^{-3}(a_2))\alpha^{-1}(h_{22}))) \\
&= \sum \alpha^2(h_{112}) \cdot a_1 \otimes \alpha^2 S(h_{111})(h_{12}(\alpha^{-2}(h_2) \cdot \alpha^{-2}(a_2))) \\
&= \sum \alpha^2(h_{112}) \cdot a_1 \otimes \alpha(S(h_{111})\alpha^{-1}(h_{12}))(\alpha^{-1}(h_2) \cdot \alpha^{-1}(a_2)) \\
&= \sum \alpha(h_{1(0)}) \cdot a_1 \otimes \alpha(h_{1(1)})(\alpha^{-1}(h_2) \cdot \alpha^{-1}(a_2)),
\end{aligned}$$

and

$$\varepsilon(h \cdot a) = \sum \varepsilon(h_1)\varepsilon(h_2)\varepsilon(a) = \varepsilon(h)\varepsilon(a).$$

Secondly

$$\begin{aligned}
&\sum \alpha(h_{212}) \otimes (h_1 \cdot b)\alpha^2(S(h_{211})\alpha^{-1}(h_{22})) \\
&= \sum \alpha(h_{212}) \otimes ((S(h_{11})\alpha^{-1}(b))\alpha(h_{12}))\alpha^2(S(h_{211})\alpha^{-1}(h_{22})) \\
&= \sum \alpha(h_{212}) \otimes [(S(h_{11})\alpha^{-1}(b))(h_{12}S\alpha(h_{211}))]\alpha^2(h_{22}) \\
&= \sum \alpha(h_{212}) \otimes [(S(h_1)b)\varepsilon(h_{211})]\alpha^2(h_{22}) \\
&= \sum h_{21} \otimes [(S(h_1)b)]\alpha^2(h_{22}) \\
&= \sum h_{12} \otimes S\alpha^2(h_{11})(bh_2) \\
&= \sum h_{12} \otimes S\alpha^2(h_{11})\varepsilon(h_{21})(b\alpha(h_{22})) \\
&= \sum \alpha(h_{112}) \otimes (S\alpha^2(h_{111})(h_{12}S(h_{21}))) (b\alpha(h_{22})) \\
&= \sum \alpha(h_{112}) \otimes [\alpha^2 S(h_{111})\alpha(h_{12})][\alpha S(h_{21})(\alpha^{-1}(b)h_{22})] \\
&= \sum \alpha(h_{112}) \otimes \alpha^2(S(h_{111})\alpha^{-1}(h_{12}))(h_2 \cdot b) \\
&= \sum h_{1(0)} \otimes \beta^2(h_{1(1)})(h_2 \cdot b),
\end{aligned}$$

Finally

$$\begin{aligned}
& \sum (h \circ k)_{(0)} \otimes (h \circ k)_{(1)} \\
&= \sum \alpha(k_{12}h_{12}) \otimes [S(h_{11})S(k_{11})][\alpha^{-1}(k_2)\alpha^{-1}(h_2)] \\
&= \sum \alpha(k_{12})\alpha(h_{12}) \otimes S\alpha(h_{11})[(S\alpha^{-1}(k_{11})\alpha^{-2}(k_2))\alpha^{-1}(h_2)] \\
&= \sum \alpha(k_{12})\alpha^2(h_{112}) \otimes [S\alpha(h_{111})(\alpha^{-1}(h_{12})\alpha^{-1}S(h_{21}))][(S\alpha^{-1}(k_{11})\alpha^{-2}(k_2))h_{22}] \\
&= \sum \alpha(k_{12})\alpha^2(h_{112}) \otimes [S\alpha(h_{111})h_{12}][S(h_{21})((S\alpha^{-2}(k_{11})\alpha^{-3}(k_2))\alpha^{-1}(h_{22}))] \\
&= \sum \alpha(k_{12})\alpha^2(h_{112}) \otimes [S\alpha(h_{111})h_{12}][(S\alpha^{-1}(h_{21})(S\alpha^{-2}(k_{11})\alpha^{-3}(k_2)))h_{22}] \\
&= \sum \alpha(k_{12})\alpha^2(h_{112}) \otimes \alpha[S(h_{111})\alpha^{-1}(h_{12})][\alpha^{-1}(h_2) \cdot \alpha^{-1}(S(k_{11})\alpha^{-1}(k_2))] \\
&= \sum \alpha(h_{1(0)}) \circ k_{(0)} \otimes \alpha(h_{1(1)})(\alpha^{-1}(h_2) \cdot \alpha^{-1}(k_{(1)})).
\end{aligned}$$

This completes the proof. \square

By the above proposition, the Hopf algebra structure on $(H * H^{op}, \alpha * \alpha)$ is given by

$$(a \# h)(b \# k) = \sum a[(S(h_{11})\beta^{-2}(b))\alpha(h_{12})] \# k\alpha(h_2),$$

$$\Delta(a \# h) = \sum a_1 \# \alpha^2(h_{112}) \otimes \beta^{-1}(a_2)(S(h_{111})\alpha^{-1}(h_{12})) \# h_2,$$

for any $a, b, h, k \in H$.

4. DRINFELD DOUBLE OF MONOIDAL HOM-HOPF ALGEBRAS

In this section, we will introduce Drinfeld double of monoidal Hom-Hopf algebras. The monoidal Hom-Hopf algebras in this section are all finite-dimensional. First we have the following definition:

Definition 4.1. Let (B, β) and (H, α) be two monoidal Hom-Hopf algebras. Then (B, H) forms a matched pair if H is a right B -Hom-module coalgebra and B is a left H -Hom-module coalgebra via

$$\triangleleft : H \otimes B \rightarrow H, \triangle : H \otimes B \rightarrow B,$$

with the compatible conditions:

- (a) $(hg) \triangleleft a = \sum (h \triangleleft (g_1 \triangleright \beta^{-1}(a_1)))(\alpha(g_2) \triangleleft a_2)$, $1_H \triangleleft a = \varepsilon_B(a)1_H$,
- (b) $h \triangleright (ab) = \sum (h_1 \triangleright \beta(a_1))((\alpha^{-1}(h_2) \triangleleft a_2) \triangleright b)$, $h \triangleright 1_B = \varepsilon_H(h)1_B$,
- (c) $\sum (h_1 \triangleleft a_1) \otimes (h_2 \triangleright a_2) = (h_2 \triangleleft a_2) \otimes (h_1 \triangleright a_1)$.

By the above definition, we have

Theorem 4.2. *Given a matched monoidal Hom-Hopf algebra pair $((B, \beta), (H, \alpha))$, we have a double cross product monoidal Hom-Hopf algebra $(B \bowtie H, \beta \otimes \alpha)$ built on the vector space $B \otimes H$ with the product*

$$(a \ltimes h)(b \ltimes g) = \sum a(h_1 \triangleright b_1) \ltimes (h_2 \triangleleft b_2)g$$

and tensor coproduct.

The antipode is defined by

$$S(a \ltimes h) = \sum S_H(h_2) \triangleright S_B(a_2) \ltimes S_H(h_1) \triangleleft S_B(a_1).$$

Proof. Firstly the unit and counit are easy to verify.

Secondly

$$\begin{aligned} & (\beta(a) \ltimes \alpha(h))((b \ltimes g)(c \ltimes f)) \\ &= \sum (\beta(a) \ltimes \alpha(h))(b(g_1 \triangleright c_1) \ltimes (g_2 \triangleleft c_2)f) \\ &= \sum \beta(a)(\alpha(h_1) \triangleright (b_1(g_{11} \triangleright c_{11}))) \ltimes (\alpha(h_2) \triangleleft (b_2(g_{12} \triangleright c_{12})))((g_2 \triangleleft c_2)f) \\ &= \sum \beta(a)[(\alpha(h_{11}) \triangleright \beta(b_{11}))((h_{12} \triangleleft b_{12}) \triangleright (g_{11} \triangleright c_{11}))] \\ & \quad \ltimes ((h_2 \triangleleft b_2) \triangleleft (\alpha(g_{12}) \triangleright \beta(c_{12})))((g_2 \triangleleft c_2)f) (b) \\ &= \sum \beta(a)[(\alpha(h_{11}) \triangleright \beta(b_{11}))((h_{12} \triangleleft b_{12}) \triangleright (\alpha^{-1}(g_1) \triangleright \beta^{-1}(c_1)))] \\ & \quad \ltimes ((h_2 \triangleleft b_2) \triangleleft (\alpha(g_{21}) \triangleright \beta(c_{21})))((\alpha(g_{22}) \triangleleft \beta(c_{22}))f) \\ &= \sum \beta(a)[(h_1 \triangleright b_1)((\alpha^{-1}(h_{21}) \triangleleft \beta^{-1}(b_{21}))\alpha^{-1}(g_1) \triangleright c_1)] \\ & \quad \ltimes [((h_{22} \triangleleft b_{22}) \triangleleft (g_{21} \triangleright c_{21}))(\alpha(g_{22}) \triangleleft \beta(c_{22}))]\alpha(f) \\ &= \sum \beta(a)[(h_1 \triangleright b_1)((\alpha^{-1}(h_{21}) \triangleleft \beta^{-1}(b_{21}))\alpha^{-1}(g_1) \triangleright c_1)] \\ & \quad \ltimes [(h_{22} \triangleleft b_{22})g_2 \triangleleft \beta(c_2)]\alpha(f) (a) \\ &= \sum (a(h_1 \triangleright b_1))((h_{21} \triangleleft b_{21})g_1 \triangleright \beta(c_1)) \ltimes [(h_{22} \triangleleft b_{22})g_2 \triangleleft \beta(c_2)]\alpha(f) (a) \\ &= \sum (a(h_1 \triangleright b_1) \ltimes (h_2 \triangleleft b_2)g)(\beta(c) \ltimes \alpha(f)) \\ &= ((a \ltimes h)(b \ltimes g))(\beta(c) \ltimes \alpha(f)). \end{aligned}$$

And

$$\begin{aligned} & \Delta(a \ltimes h)\Delta(b \ltimes g) \\ &= \sum ((a_1 \ltimes h_1) \otimes (a_2 \ltimes h_2))((b_1 \ltimes g_1) \otimes (b_2 \ltimes g_2)) \\ &= \sum a_1(h_{11} \triangleright b_{11}) \ltimes (h_{12} \triangleleft b_{12})g_1 \otimes a_2(h_{21} \triangleright b_{21}) \ltimes (h_{22} \triangleleft b_{22})g_2 \\ &= \sum a_1(\alpha^{-1}(h_1) \triangleright \beta^{-1}(b_1)) \ltimes (\alpha(h_{211}) \triangleleft \beta(b_{211}))g_1 \\ & \quad \otimes a_2(\alpha(h_{212}) \triangleright \beta(b_{212})) \ltimes (h_{22} \triangleleft b_{22})g_2 \end{aligned}$$

$$\begin{aligned}
&= \sum a_1(\alpha^{-1}(h_1) \triangleright \beta^{-1}(b_1)) \ltimes (\alpha(h_{212}) \triangleleft \beta(b_{212})) g_1 \\
&\quad \otimes a_2(\alpha(h_{211}) \triangleright \beta(b_{211})) \ltimes (h_{22} \triangleleft b_{22}) g_2 \quad (c) \\
&= \sum a_1(h_{11} \triangleright b_{11}) \ltimes (h_{21} \triangleleft b_{21}) g_1 \otimes a_2(h_{12} \triangleright b_{12}) \ltimes (h_{22} \triangleleft b_{22}) g_2 \\
&= \Delta((a \ltimes h)(b \ltimes g)).
\end{aligned}$$

Obviously $\varepsilon = \varepsilon_B \otimes \varepsilon_H$ is a monoidal Hom-algebra map.

Finally it is straightforward to check that $(\beta \otimes \alpha)((a \ltimes h)(b \ltimes g)) = (\beta \otimes \alpha)(a \ltimes h)(\beta \otimes \alpha)(b \ltimes g)$ and $\Delta \circ (\beta \otimes \alpha) = [(\beta \otimes \alpha) \otimes (\beta \otimes \alpha)] \circ \Delta$. Thus $B \bowtie H$ is a monoidal Hom-bialgebra. It is easy to verify that $S * id = id * S = (1_B \ltimes 1_H)\varepsilon$.

This completes the proof. \square

Proposition 4.3. *In the bicrossproduct $(B \# H, \beta \otimes \alpha)$ defined as before, if we define $\triangleright : B^* \otimes H \rightarrow H$ by*

$$f \triangleright h = \sum \langle f, \beta(h_{(1)}) \rangle \alpha^2(h_{(0)}),$$

and $\triangleleft : B^ \otimes H \rightarrow B^*$ by*

$$\langle f \triangleleft h, a \rangle = \sum \langle f, \alpha^{-1}(h) \cdot \beta^{-2}(a) \rangle.$$

Then $(B^, (\beta^{-1})^*)$ is a right (H, α) -Hom module coalgebra, and (H, α) is a left $(B^*, (\beta^{-1})^*)$ -Hom module coalgebra.*

Proof. The proof is straightforward. \square

Proposition 4.4. *Under the action defined in the above proposition, $(H \bowtie B^*, \alpha \otimes (\beta^{-1})^*)$ is a matched monoidal Hom-Hopf algebra pair. The product and antipode are given by*

$$(h \ltimes f)(k \ltimes g) = \sum h \alpha^2(k_{1(0)}) \ltimes \langle f, \beta(k_{1(1)}) (\alpha^{-1}(k_2) \cdot \beta^{-2}(?)) \rangle g,$$

$$S(h \ltimes f) = \sum S_B^*(f_2) \triangleright S_H(h_2) \ltimes S_B^*(f_1) \triangleleft S_H(h_1),$$

for any $h, k \in H$ and $f, g \in B^$.*

Proof. We need only to verify the compatible conditions. Firstly for any $a, b \in B$, $h, k \in H$, and $f, g \in B^*$,

$$\begin{aligned}
& \sum < f \triangleleft (g_1 \triangleright \alpha^{-1}(h_1)), a_1 > < (\beta^{-1})^*(g_2) \triangleleft h_2, a_2 > \\
&= \sum < f, \alpha^{-1}((g_1 \triangleright \alpha^{-1}(h_1))) \cdot \beta^{-2}(a_1) > < (\beta^{-1})^*(g_2) \triangleleft h_2, a_2 > \\
&= \sum < f, \alpha^{-1}((g_1 \triangleright \alpha^{-1}(h_1))) \cdot \beta^{-2}(a_1) > < (\beta^{-1})^*(g_2), \alpha^{-1}(h_2) \cdot \beta^{-2}(a_2) > \\
&= \sum < f, (\beta^*(g_1) \triangleright \alpha^{-2}(h_1)) \cdot \beta^{-2}(a_1) > < g_2, \alpha^{-2}(h_2) \cdot \beta^{-3}(a_2) > \\
&= \sum < f, h_{1(0)} \cdot \beta^{-2}(a_1) > < g_1, h_{1(1)} > < g_2, \alpha^{-2}(h_2) \cdot \beta^{-3}(a_2) > \\
&= \sum < f, h_{1(0)} \cdot \beta^{-2}(a_1) > < g, h_{1(1)}(\alpha^{-2}(h_2) \cdot \beta^{-3}(a_2)) > \\
&= \sum < f, (\alpha^{-1}(h) \cdot \beta^{-2}(a))_1 > < g, (\alpha^{-1}(h) \cdot \beta^{-2}(a))_2 > \\
&= < (f * g) \triangleleft h, a > .
\end{aligned}$$

Then

$$\begin{aligned}
f \triangleright (hk) &= \sum < f, \beta((hk)_{(1)}) > \alpha^2((hk)_{(0)}) \\
&= \sum < f, \beta^2(h_{1(1)})(g_2 \cdot k_{(1)}) > \alpha^3(h_{1(0)})\alpha^2(k_{(0)}) \\
&= \sum < f_1, \beta^2(h_{1(1)}) > < f_2, h_2 \cdot k_{(1)} > \alpha^3(h_{1(0)})\alpha^2(k_{(0)}) \\
&= \sum (f_1 \triangleright \beta(h))((\beta^*(f_2) \triangleleft h_2) \triangleright k).
\end{aligned}$$

Finally

$$\begin{aligned}
& \sum < f_1 \triangleleft h_1, a > < g, f_2 \triangleright h_2 > \\
&= \sum < f_1, \alpha^{-1}(h_1) \cdot \beta^{-2}(a) > < f_2, \beta(h_{2(1)}) > < g, \alpha^2(h_{2(0)}) > \\
&= \sum < f, (\alpha^{-1}(h_1) \cdot \beta^{-2}(a))\beta(h_{2(1)}) > < g, \alpha^2(h_{2(0)}) > \\
&= \sum < f_2, \alpha^{-1}(h_2) \cdot \beta^{-2}(a) > < f_1, \beta(h_{1(1)}) > < g, \alpha^2(h_{1(0)}) > \\
&= \sum < f_2 \triangleleft h_2, a > < g, f_1 \triangleright h_1 > .
\end{aligned}$$

This completes the proof. \square

Corollary 4.5. *For the bicrossproduct monoidal Hom-Hopf algebra $(H * H^{op}, \alpha * \alpha)$ in the Proposition 3.3, we have the Drinfeld double $\mathcal{D}(H) = H^{op} \bowtie H^*$. Moreover the product is given by*

$$(h \bowtie f)(k \bowtie g) = \sum \alpha^2(k_{21})h \bowtie < f, S(k_1)(\alpha^{-2}(?)k_{22}) > g$$

for any $h, k \in H$ and $f, g \in H^*$.

Proof. By the above proposition, it is straightforward to get this result. \square

Definition 4.6. A quasitriangular monoidal Hom-Hopf algebra is a monoidal Hom-Hopf algebra (H, α) with an element $R \in H \otimes H$ satisfying

- (a) $(\varepsilon \otimes id)R = (id \otimes \varepsilon)R = 1$,
- (b) $\Delta^{op}(x)R = R\Delta(x)$ for any $x \in H$,
- (c) $(\Delta \otimes \alpha^{-1})R = R^{13}R^{23}$,
- (d) $(\alpha^{-1} \otimes \Delta)R = R^{13}R^{12}$.

Remark 4.7. Note that the above definition is different from Definition 2.7 in [8].

In the above definition, we will denote $R = \sum R^{(1)} \otimes R^{(2)}$ throughout this section. Therefore we could rewrite the equalities (c) and (d) as follows:

$$(\Delta \otimes id)R = \sum \alpha(R^{(1)}) \otimes \alpha(r^{(1)}) \otimes \alpha(R^{(2)}r^{(2)}), \quad (4.1)$$

$$(id \otimes \Delta)R = \sum \alpha(R^{(1)}r^{(1)}) \otimes \alpha(r^{(2)}) \otimes \alpha(R^{(2)}). \quad (4.2)$$

Example 4.8. Let H be the monoidal Hom-algebra generated by the elements $1_H, g, x$ satisfying the following relations:

$$1_H 1_H = 1_H, \quad 1_H g = g 1_H = g, \quad 1_H x = x 1_H = -x,$$

$$g^2 = 1_H, \quad x^2 = 0, \quad gx = -xg.$$

The automorphism $\alpha : H \rightarrow H$ is defined by

$$\alpha(1_H) = 1_H, \quad \alpha(g) = g, \quad \alpha(x) = -x, \quad \alpha(gx) = -gx.$$

Then (H, α) is a Hom-associative algebra, and $\alpha^2 = id$.

Define

$$\Delta(1_H) = 1_H \otimes 1_H, \quad \Delta(g) = g \otimes g,$$

$$\Delta(x) = (-x) \otimes g + 1 \otimes (-x),$$

$$\varepsilon(1_H) = 1, \quad \varepsilon(g) = 1, \quad \varepsilon(x) = 0,$$

$$S(1_H) = 1_H, \quad S(g) = g, \quad S(x) = -gx.$$

Then (H, α) is a monoidal Hom-Hopf algebra. Let $R = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g)$.

It is easy to check that (H, α, R) is quasitriangular.

Proposition 4.9. *Let (H, α) be a quasitriangular monoidal Hom-Hopf algebra with the quasitriangular structure R . Then R satisfies the QHYBEs*

$$R^{12}(R^{13}R^{23}) = (R^{13}R^{23})R^{12} \quad (4.3)$$

and

$$(R^{12}R^{13})R^{23} = R^{23}(R^{13}R^{12}). \quad (4.4)$$

Proof. The proof is essentially the same as in the Hopf algebra setting. \square

Proposition 4.10. *In the Drinfeld double $(H^{op} \bowtie H^*, \alpha \otimes (\alpha^{-1})^*)$, the quasitriangular structure*

$$R = \sum_i 1 \bowtie h_i^* \bowtie S^{-1}(h_i) \bowtie \varepsilon,$$

where $\{h_i\}$ and $\{h_i^*\}$ are a base of H and its dual base in H^* respectively.

Proof. Firstly, obviously $(\varepsilon \otimes id)R = (id \otimes \varepsilon)R = 1$. Then for any $h \bowtie f \in H^{op} \bowtie H^*$, and $l, g \in H^*$, $a, b \in H$, on one hand

$$\begin{aligned} & \langle \Delta^{op}(h \bowtie f)R, g \otimes a \otimes l \otimes b \rangle \\ &= \sum \langle (h_2 \bowtie f_2 \otimes h_1 \bowtie f_1)R, g \otimes a \otimes l \otimes b \rangle \\ &= \sum \langle \alpha(h_2) \bowtie f_2 h_i^* \otimes \alpha^2(S^{-1}(h_i)_{21})h_1 \bowtie f_1(S(S^{-1}(h_i)_1)(\alpha^{-3}(?)S^{-1}(h_i)_{22})), \\ & \quad g \otimes a \otimes l \otimes b \rangle \\ &= \sum \langle g, \alpha(h_2) \rangle \langle f_2 h_i^*, a \rangle \langle l, \alpha^2(S^{-1}(h_i)_{21})h_1 \rangle \\ & \quad \langle f_1, S(S^{-1}(h_i)_1)(\alpha^{-3}(b)S^{-1}(h_i)_{22}) \rangle \\ &= \sum \langle g, \alpha(h_2) \rangle \langle f_2, a_2 \rangle \langle h_i^*, a_2 \rangle \langle l, \alpha^2(S^{-1}(h_i)_{21})h_1 \rangle \\ & \quad \langle f_1, S(S^{-1}(h_i)_1)(\alpha^{-3}(b)S^{-1}(h_i)_{22}) \rangle \\ &= \sum \langle g, \alpha(h_2) \rangle \langle l, \alpha^2(S^{-1}(a_{212}))h_1 \rangle \langle f, [a_{22}(\alpha^{-3}(b)S^{-1}(a_{211}))]a_1 \rangle \\ &= \sum \langle g, \alpha(h_2) \rangle \langle l, \alpha^2(S^{-1}(a_{212}))h_1 \rangle \langle f, [a_{22}\alpha^{-2}(b)][S^{-1}\alpha(a_{211})\alpha^{-1}(a_1)] \rangle \\ &= \sum \langle g, \alpha(h_2) \rangle \langle l, S^{-1}(a_1)h_1 \rangle \langle f, a_2\alpha^{-1}(b) \rangle, \end{aligned}$$

and on the other hand

$$\begin{aligned} & \langle R\Delta(h \bowtie f), g \otimes a \otimes l \otimes b \rangle \\ &= \sum \langle (1 \bowtie h_i^*)(h_1 \bowtie f_1) \otimes (S^{-1}(h_i) \bowtie \varepsilon_H)(h_2 \bowtie f_2), g \otimes a \otimes l \otimes b \rangle \\ &= \sum \langle \alpha^3(h_{121}) \bowtie h_i^*(S(h_{11})(\alpha^{-2}(?)h_{122}))f_1 \otimes h_2 S^{-1}(h_i) \bowtie f_2 \circ \alpha^{-1}, \\ & \quad g \otimes a \otimes l \otimes b \rangle \\ &= \sum \langle g, \alpha^3(h_{121}) \rangle \langle h_i^*, S(h_{11})(\alpha^{-2}(a_1)h_{122}) \rangle \langle f_1, a_2 \rangle \langle l, h_2 S^{-1}(h_i) \rangle \\ & \quad \langle f_2, \alpha^{-1}(b) \rangle \\ &= \sum \langle g, \alpha^3(h_{121}) \rangle \langle f, a_2\alpha^{-1}(b) \rangle \langle l, h_2[(S^{-1}(h_{122})S^{-1}(\alpha^{-2}(a_1)))h_{11}] \rangle \\ &= \sum \langle g, \alpha^2(h_{21}) \rangle \langle f, a_2\alpha^{-1}(b) \rangle \langle l, (\alpha(h_{222})S^{-1}\alpha(h_{221}))(S^{-1}\alpha^{-1}(a_1)\alpha^{-1}(h_1)) \rangle \\ &= \sum \langle g, \alpha(h_2) \rangle \langle f, a_2\alpha^{-1}(b) \rangle \langle l, (S^{-1}(a_1)h_1) \rangle, \end{aligned}$$

Hence $R\Delta(h \bowtie f) = \Delta^{op}(h \bowtie f)R$.

Similarly we have $(\Delta \otimes \alpha^{-1})R = R^{13}R^{23}$, $(\alpha^{-1} \otimes \Delta)R = R^{13}R^{12}$.

The proof is completed. \square

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